

STEINER SYMMETRALS AND THEIR DISTANCE FROM A BALL

BY

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ABSTRACT

It is known that given any convex body $K \subset \mathbb{R}^n$ there is a sequence of suitable iterated Steiner symmetrizations of K that converges, in the Hausdorff metric, to a ball of the same volume. Hadwiger and, more recently, Bourgain, Lindenstrauss and Milman have given estimates from above of the number N of symmetrizations necessary to transform K into a body whose distance from the equivalent ball is less than an arbitrary positive constant.

In this paper we will exhibit some examples of convex bodies which are “hard to make spherical”. For instance, for any choice of positive integers $n \geq 2$ and m , we construct an n -dimensional convex body with the property that any sequence of m symmetrizations does not decrease its distance from the ball. A consequence of these constructions are some lower bounds on the number N .

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1. Introduction

Let K be an n -dimensional convex body. It is known (see, e.g., [H1]) that there is a sequence of suitable iterated Steiner symmetrizations of K that converges, in the Hausdorff metric, to a ball of the same volume.

In this paper we will study quantitative information on this convergence.

Let \mathcal{K}_1 denote the class of n -dimensional convex bodies whose volume equals κ_n , the volume of the unit ball B_n . For any $\varepsilon > 0$ we denote by $N(n, \varepsilon)$ the minimum number of successive Steiner symmetrizations needed to transform any body in \mathcal{K}_1 into one whose Hausdorff distance from B_n is less than or equal to ε (see the next section for definitions).

Hadwiger [H2] obtained an upper bound for $N(n, \varepsilon)$: in the subclass of \mathcal{K}_1 of the bodies contained in a ball of radius R ,

$$(1.1) \quad N(n, \varepsilon) \leq (4R\sqrt{n} + 2\varepsilon)^n \varepsilon^{-2n}.$$

In the proof Hadwiger uses some estimates of the number of balls needed to cover a given cube; substituting them with better estimates now available in the literature (see, for example, [FTK]) one can improve the previous bound to $N(n, \varepsilon) \leq (4R)^n (n \log n + n \log \log n + 4n) \varepsilon^{-2n}$.

More recently, Bourgain, Lindenstrauss and Milman [BLM] have shown that there exist universal constants a and c such that

$$(1.2) \quad N(n, a) \leq cn \log n.$$

Regarding $N(n, \varepsilon)$ for ε small they prove that

$$(1.3) \quad S_{v_m} \dots S_{v_1}(K) \subset (1 + \varepsilon)B_n,$$

with $m \leq cn \log n + f(\varepsilon)n$, for some function f of ε (S_{v_i} denote suitable Steiner symmetrizations), where n is larger than a given function of ε that grows unboundedly as $\varepsilon \rightarrow 0$. The asymptotic behaviour of $f(\varepsilon)$ as $\varepsilon \rightarrow 0$ is $f(\varepsilon) \approx e^{|\log \varepsilon|/\varepsilon^2}$. Their paper does not contain results regarding the other inclusion, that of $(1 - \varepsilon)B_n$ in $S_{v_m} \dots S_{v_1}(K)$.

Finally, we mention that Tsolomitis [T] has studied similar problems for some generalizations of the Steiner symmetrization.

In this paper we seek lower bounds on the function $N(n, \varepsilon)$. These bounds will be consequences of the existence of some convex bodies which are “hard to make spherical”.

THEOREM 1: *Let $n \geq 2$. Given any positive integer m , there exists a non-spherical origin-symmetric convex body H in \mathbb{R}^n such that for any choice of m directions v_1, v_2, \dots, v_m the body $S_{v_m} \dots S_{v_1}(H)$ has the same inner and outer radius as H .*

We recall that the inner radius r and outer radius R of a convex body H are respectively the largest radius of a ball contained in H and the smallest radius of a ball containing H . When the body is origin symmetric these balls are centered at the origin and the Hausdorff distance from H to B_n is $\max\{1 - r, R - 1\}$.

An immediate consequence of Theorem 1 is that it is not possible to reduce the Hausdorff distance of H from the ball of the same volume using at most m symmetrizations.

COROLLARY 2: *Let $n \geq 2$. Given any positive integer m there exists a non-spherical origin-symmetric body H in \mathbb{R}^n of volume κ_n such that for any choice of m directions v_1, v_2, \dots, v_m the body $S_{v_m} \dots S_{v_1}(H)$ has the same Hausdorff distance from B_n as H .*

Estimates of the distance from B_n of these, or similar, bodies give estimates of $N(n, \varepsilon)$ from below.

THEOREM 3: *Let $N(n, \varepsilon)$ be the function defined in (2.1) and let $n \geq 2$. Then*

$$(1.4) \quad N(n, \varepsilon) \geq \frac{\log(\log(1/\varepsilon))}{\log 2} (1 + o(1)) \quad \text{as } \varepsilon \rightarrow 0.$$

When n is large compared to m , bodies like those in Corollary 2 can be easily constructed. Let $n \geq m + 2$ and let B_{n-1} be the intersection of the unit ball B_n with a hyperplane through the origin. Let $R > 2$ be arbitrary and let H be the convex hull of RB_{n-1} and rB_n , with $0 < r < R$ such that $\text{vol}(H) = \kappa_n$. The Hausdorff distance of H from B_n is not reduced by any m symmetrizations (since symmetrizations with respect to directions v_1, \dots, v_m leave the points in $RB_{n-1} \cap v_1^\perp \cap \dots \cap v_m^\perp$ unchanged, where $v^\perp = \{x \in \mathbb{R}^n : \langle x, v \rangle = 0\}$). This distance is $R - 1$ and so it can be made arbitrarily large. Therefore

$$(1.5) \quad N(n, \varepsilon) \geq n - 1 \quad \forall \varepsilon > 0.$$

Finally, we would like to mention a technical result which might be of interest in itself. Proposition 4, in a particular case, implies the following statement. Given $v \in S^{n-1}$ we denote by π_v the reflection with respect to v^\perp .

PROPOSITION 4 (in the case of two directions): *Let $n \geq 2$. There exist subsets A of S^{n-1} , with arbitrarily small measure, such that for any choice of the directions $v_1, v_2 \in S^{n-1}$,*

$$(A \cap \pi_{v_1}(A)) \cap \pi_{v_2}(A \cap \pi_{v_1}(A)) \neq \emptyset.$$

This result is related to various problems, for instance to additive properties of sequences of integers studied by P. Erdős (see [HR], Ch. 3).

2. Preliminaries

Given a direction $v \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, **Steiner symmetrization along v** is the mapping that associates to each convex body $K \subset \mathbb{R}^n$ the unique convex body $S_v(K)$ with the following properties: given any line l parallel to v , $K \cap l$ and $S_v(K) \cap l$ are either both empty or are segments of the same length, and $S_v(K) \cap l$ is symmetric with respect to v^\perp .

Let $d_H(\cdot, \cdot)$ denote the Hausdorff distance. We define

$$(2.1) \quad N(n, \varepsilon) = \sup_{K \in \mathcal{K}_1} \inf \{m \in \mathbb{N} : \exists v_1, \dots, v_m \in S^{n-1} \text{ with } d_H(B_n, S_{v_m} \dots S_{v_1}(K)) \leq \varepsilon\}.$$

We now describe the ideas behind the construction of the body H in Theorem 1 in the simple case of an origin-symmetric convex body K such that any single symmetrization does not decrease its outer radius.

For any $z \in \mathbb{R}^n$ and for any direction $v \in S^{n-1}$, let $\pi_v(z) = z - 2\langle z, v \rangle v$. If R denotes the outer radius of K ,

$$(2.2) \quad z \in S_v(K) \cap RS^{n-1} \quad \text{if and only if} \quad z, \pi_v(z) \in K \cap RS^{n-1}.$$

No symmetrization decreases the outer radius of the compact set K if and only if $S_v(K) \cap RS^{n-1} \neq \emptyset$ for any direction v . Thus the required property can be rephrased in terms which involve only the intersection of K with the outer sphere. Let A be an origin-symmetric subset of S^{n-1} such that

$$(2.3) \quad \text{for any direction } v \text{ there exists } z \in A \text{ such that } \pi_v(z) \in A.$$

Then no symmetrization decreases the outer radius of its convex hull. To construct a convex body whose outer and inner radius remain unchanged, it suffices to construct two closed disjoint subsets A and B of S^{n-1} that satisfy (2.3) and to consider the convex hull of rS^{n-1} and RB for some $0 < r < R$, with R/r so close to 1 that the boundary of this convex set contains rA .

If its $(n - 1)$ -dimensional Hausdorff measure is large enough, any subset of S^{n-1} satisfies (2.3), but we are interested mainly in subsets whose measure is small. Figure 1 shows some subsets of S^1 that satisfy (2.3).

The set A is $[0, \pi/2] \cup [\pi, 3\pi/2]$; the set B is obtained by removing the middle third of each interval of A , as in a Cantor set construction. Note that one can iterate this procedure and obtain sets of arbitrarily small measure that still satisfy (2.3), and this property also passes to the limit. The set C in Fig. 1 is defined by $C = \{\pi/4, 5/4\pi\} \cup [7/12\pi, 11/12\pi] \cup [19/12\pi, 23/12\pi]$. Note that B and C are disjoint.

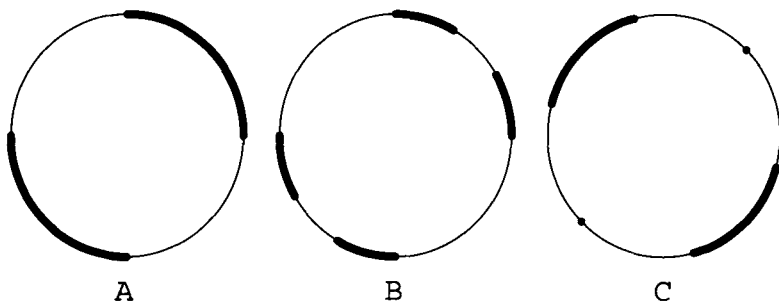


Figure 1

Given an ordered set V of m directions v_1, v_2, \dots, v_m in S^{n-1} , for any $x \in \mathbb{R}^n$ we call the **orbit** of x the set $O_V(x)$ defined by

$$O_V(x) = \{x, \pi_{v_j}(\pi_{v_{j-1}}(\dots \pi_{v_1}(x))), \text{ for any } m \geq j_1 > j_2 > \dots > j_s \geq 1\}.$$

If r and R are respectively the inner and outer radius of the origin symmetric convex body K , then

$$z \in S_{v_m} \dots S_{v_1}(K) \cap RS^{n-1} \text{ if and only if } O_V(z) \subset K \cap RS^{n-1}$$

and

$$z \in \partial(S_{v_m} \dots S_{v_1}(K)) \cap rS^{n-1} \text{ if and only if } O_V(z) \subset \partial K \cap rS^{n-1}.$$

To construct the set H of Theorem 1 we will construct subsets A of S^{n-1} with the property that

$$(2.4) \quad \begin{aligned} &\text{for any choice of the sequence } V \text{ of } m \text{ directions } v_1, v_2, \dots, v_m \\ &\text{there exists } x \in A \text{ such that } O_V(x) \subset A. \end{aligned}$$

We call this property the **m -orbit property**.

PROPOSITION 4: *For any positive integers $n \geq 2$ and m there exist subsets of S^{n-1} of arbitrarily small \mathcal{H}^{n-1} -measure which satisfy the m -orbit property.*

Here \mathcal{H}_{n-1} stands for $(n-1)$ -dimensional Hausdorff measure.

3. A discretization of the problem

In order to construct the sets $K \cap rS^{n-1}$ and $K \cap RS^{n-1}$, we solve a similar problem in the discrete case. We use an argument introduced by Erdős in the study of additive properties of sequences (see [HR], Ch. 3). In the planar case the 1-orbit property has the following discretized version: given a positive integer M , $A + A = \{0, \dots, M-1\} \bmod M$. Erdős studied the problem of finding “small” increasing subsequences A of \mathbb{N} such that $A + A = \mathbb{N}$ and gave a probabilistic proof of the existence of such subsequences.

If $x, y \in S^{n-1}$ we denote by $d(x, y)$ their **geodesic distance**. As usual, we write $d(x, A)$ for $\inf_{a \in A} d(x, a)$. A finite subset T of S^{n-1} such that

$$(3.1) \quad \inf_{x \in T} d(x, T \setminus \{x\}) \geq 2\delta, \quad \sup_{y \in S^{n-1}} d(y, T) < 2\delta, \text{ for a given } \delta > 0,$$

is called a **δ -net**. Let t denote the cardinality of T . Formula (3.1) implies that the spherical caps centred in T and with radius δ are disjoint. Computing the volumes one gets

$$(3.2) \quad \delta \leq 2 \left(\frac{n\kappa_n}{\kappa_{n-1}t} \right)^{\frac{1}{n-1}}.$$

For each $v \in T$ we define a map $\pi_v' : T \rightarrow T$, the map which corresponds in this setting to the reflection with respect to v^\perp , as follows. To every $z \in T$ we associate a point w of T that minimizes the distance of $\pi_v(z)$ from T . From (3.1), we have $d(\pi_v'(z), \pi_v(z)) < 2\delta$.

The map π_v' is not necessarily injective, but the cardinality of $\pi_v'^{-1}(x)$ is bounded by 3^n for δ small enough. This follows from the observation that if k points of T have the same image under the map π_v' , then there are k disjoint spherical caps of radius δ contained in a spherical cap of radius 3δ .

Fix $m \in \mathbb{N}$. For every sequence $W = w_1, w_2, \dots, w_m$ of m directions in T we define the discretized W -orbit of a point $x \in T$ as follows:

$$O'_W(x) = \{x, \pi_{w_{j_s}}'(\pi_{w_{j_{s-1}}}(\dots \pi_{w_{j_1}}'(x))), \text{ with } m \geq j_1 > j_2 > \dots > j_s \geq 1\}.$$

Notice that $|O'_W(x)| \leq 2^m$.

LEMMA 5: For every $W \in T^m$, there exist at least $\lfloor t/3^{m(n+1)} \rfloor$ disjoint discretized W -orbits.

Proof: For every $x \in T$, consider the sets

$$E(x) = \{y \in T : O'_W(x) \cap O'_W(y) \neq \emptyset\}.$$

Let us estimate the cardinality of $E(x)$. An element y belongs to $E(x)$ if and only if there are indices $m \geq i_1 > i_2 > \dots > i_q \geq 1$, $m \geq j_1 > j_2 > \dots > j_s \geq 1$ such that

$$y \in \pi'_{w_{i_1}}(\pi'^{-1}_{w_{i_2}}(\dots \pi'^{-1}_{w_{i_q}}(\pi'_{w_{j_s}}(\pi'_{w_{j_{s-1}}}(\dots \pi'_{w_{j_1}}(x)))))).$$

Thus $|E(x)| \leq 2^m(3^n + 1)^m < 3^{m(n+1)}$, where 3^n comes from the bound on the cardinality of $\pi'^{-1}_v(x)$. Let q and x_1, x_2, \dots, x_q be such that

$$x_i \notin E(x_1) \cup E(x_2) \cup \dots \cup E(x_{i-1}), \quad \forall i$$

and $T = \bigcup_{i=1}^q E(x_i)$. By definition $O'_W(x_i) \cap O'_W(x_j) = \emptyset$ for every $i \neq j$. The bound $t \leq q3^{m(n+1)}$ concludes the proof. ■

Now our aim is to construct a small subset of T such that, for every $W \in T^m$, it contains a W -orbit. We shall use a probabilistic argument.

Let p_1, p_2, \dots, p_t denote the points of T , fix $\alpha \in (0, 1)$ and let Γ be the set of the subsets of T . We define a probability measure \mathcal{P} on Γ by $\mathcal{P}(A) = \alpha^{|A|}(1-\alpha)^{t-|A|}$ for $A \in \Gamma$.

If we identify the set A with the element $(a_1, a_2, \dots, a_t) \in \Omega = \{0, 1\}^t$ such that $a_i = 1$ if $p_i \in A$ and $a_i = 0$ otherwise, then $(\Omega, \mathcal{G}, \mathcal{P})$, where \mathcal{G} is the set of subsets of Ω , is the Bernoulli probability with parameter α . The probability that a random subset A of T contains a fixed $B \in \Gamma$ is

$$\mathcal{P}(A \in \Gamma : A \supset B) = \sum_{A \supset B} \mathcal{P}(A) = \alpha^{|B|}.$$

LEMMA 6: Let T be a δ -net and let $t = |T|$. Let f be an integer in $[1, t]$. The \mathcal{P} -probability that a subset A of T has $|A| < f$ and contains a W -orbit for every $W \in T^m$ is greater than or equal to

$$(3.3) \quad 1 - t^m(1 - \alpha^{2^m})^{\lfloor t/3^{m(n+1)} \rfloor} - \sum_{i=f}^t \binom{t}{i} \alpha^i (1 - \alpha)^{t-i}.$$

Proof: By Lemma 5, there exist x_1, x_2, \dots, x_q such that $O'_W(x_i) \cap O'_W(x_j) = \emptyset$ for $i \neq j$, and $q \geq [t/3^{m(n+1)}]$. Hence, for a fixed W , we can write

$$\begin{aligned} \mathcal{P}(\{A : A \not\supseteq O'_W(x), \forall x \in T\}) &\leq \mathcal{P}(\cap_{i=1}^q \{A : A \not\supseteq O'_W(x_i)\}) \\ &= \prod_{i=1}^q \mathcal{P}(\{A : A \not\supseteq O'_W(x_i)\}) \leq \prod_{i=1}^q (1 - \alpha^{2^m}) \leq (1 - \alpha^{2^m})^{[t/3^{m(n+1)}]}, \end{aligned}$$

where we have used the fact that all the events $\{A \in \Gamma : A \supset O'_W(x_i)\}$ are independent since the orbits $O'_W(x_i)$ are disjoint.

We can now write

$$\begin{aligned} &\mathcal{P}\left(\bigcup_{W \in T^m} \{A : A \not\supseteq O'_W(x) \forall x \in T\} \cup \{A : |A| \geq f\}\right) \\ &\leq \mathcal{P}(\{A : A \not\supseteq O'_W(x), \forall x \in T\}) + \mathcal{P}(\{A : |A| \geq f\}) \\ &\leq t^m (1 - \alpha^{2^m})^{[t/3^{m(n+1)}]} + \sum_{i=f}^t \binom{t}{i} \alpha^i (1 - \alpha)^{t-i}. \quad \blacksquare \end{aligned}$$

4. Existence results

Given $A \subset S^{n-1}$ and $a > 0$ we denote by $(A)_a = \{x \in S^{n-1} : d(x, A) < a\}$ the open a -neighborhood of A .

LEMMA 7: *Let T be a δ -net, and let A' be a subset of T that contains a discretized W -orbit for any $W \in T^m$. Then $(A')_{6m\delta}$ satisfies the m -orbit property.*

Proof: Let $V = (v_1, v_2, \dots, v_m)$ be any sequence of m directions in S^{n-1} , for each i let $w_i \in T$ be such that $d(v_i, w_i) < 2\delta$ and let $W = (w_1, w_2, \dots, w_m)$. Then for any $y \in T$ we have $d(\pi_{v_i}(y), \pi_{w_i}'(y)) < 6\delta$. There exists $x \in A'$ such that $O'_W(x) \subset A'$; it is straightforward to check that $O_V(x) \subset (A')_{6m\delta}$. \blacksquare

LEMMA 8: *Let $A \subset S^{n-1}$ be such that $\mathcal{H}_{n-1}(S^{n-1} \setminus A) < n\kappa_n/2^m$. Then A satisfies the m -orbit property.*

Proof: Let $B = S^{n-1} \setminus A$ and let $\overleftarrow{V} = (v_m, v_{m-1}, \dots, v_1)$ be V in the opposite order. The set $O_{\overleftarrow{V}}(B) = \bigcup_{y \in B} O_{\overleftarrow{V}}(y)$ is the union of at most 2^m sets obtained from B through a finite number of reflections and thus of sets which have the same measure as B . This implies that $\mathcal{H}_{n-1}(O_{\overleftarrow{V}}(B)) < n\kappa_n$ and thus that $S^{n-1} \setminus O_{\overleftarrow{V}}(B) \neq \emptyset$. If $x \in S^{n-1} \setminus O_{\overleftarrow{V}}(B)$ then $O_V(x) \subset S^{n-1} \setminus B = A$. \blacksquare

Proof of Proposition 4: Let $T \subset S^{n-1}$ be a δ -net and let $t = |T|$. Let $\alpha = t^{-\beta}$, for some $0 < \beta < 1/2^m$ and $f = \alpha t$. If t is large enough then the probability in (3.3) is positive. Therefore, by Lemma 6, there exists a subset A' of T which contains a W -orbit for each $W \subset T^m$. Let $A = (A')_{6m\delta}$; by Lemma 7 it satisfies the m -orbit property.

We have that $\mathcal{H}_{n-1}(A) \leq |A'| \kappa_{n-1}(6m\delta)^{n-1} \leq t^{1-\beta} \kappa_{n-1}(6m\delta)^{n-1}$ which, due to (3.2), becomes arbitrarily small as $\delta \rightarrow 0$. ■

Proof of Theorem 1: Let A be a subset of S^{n-1} with the m -orbit property and $\mathcal{H}_{n-1}(A) \leq n\kappa_n/2^{m+1}$. Without loss of generality we may assume that A is origin symmetric. By continuity it is possible to choose a positive number a such that

$$(4.1) \quad \mathcal{H}_{n-1}((A)_a) \leq n\kappa_n/2^m.$$

We claim that if the two positive numbers r, R are such that $r/R < \cos a$ then, defining H as the convex hull of rS^{n-1} and of RA , we have that H is origin symmetric, that its inner and outer radius are respectively r and R , that $H \cap RS^{n-1} = RA$ and $\partial H \supset rB$, where $B = S^{n-1} \setminus (A)_a$. The only assertion that needs to be proved is that $\partial H \supset rB$. Let $x \in rB$ and let π be a hyperplane supporting rS^{n-1} at x . The intersection of this hyperplane with RS^{n-1} is a spherical cap with radius $\arccos(r/R)$. Therefore, for our choice of r and R , the spherical cap cut by π on RS^{n-1} does not contain points of RA . This implies that π supports H .

As observed in Section 2 the set H satisfies the claim of Theorem 1 if and only if both the intersections of ∂H with the boundary of the outer and inner spheres satisfy the m -orbit property. This is true for the outer sphere by construction, while for the inner sphere it follows from Lemma 8 and (4.1). ■

Remark: As is clear from the previous proof, a is a “measure” of the distance of H from the equivalent sphere. This constant depends on n and m . In the next section we try to maximize this distance for bodies similar to H .

5. Lower bounds

If $\varepsilon(n, m)$ denotes the Hausdorff distance from B_n of the body $H \in \mathcal{K}_1$ constructed in the proof of Theorem 1, then $N(n, \varepsilon(n, m)) > m$. Thus, estimates of $\varepsilon(n, m)$ provide a lower bound for $N(n, \varepsilon)$. However, we can get better estimates if we relax the hypothesis that both the outer and inner radii remain unchanged.

We shall construct convex bodies in \mathcal{K}_1 so that m arbitrary Steiner symmetrizations do not reduce their outer radii and furthermore their Hausdorff distance from B_n is attained by the outer ball. In order to do this we need the following lemma.

LEMMA 9: *Suppose $rB_n \subset K \subset RB_n$. If $\mathcal{H}_{n-1}(\partial K \cap rS^{n-1}) \geq \frac{1}{2}\mathcal{H}_{n-1}(rS^{n-1})$ and $\text{vol}(cB_n) = \text{vol}(K)$, then $c \leq (r + R)/2$.*

Proof: Let C be the convex body defined as the union of all points in RB_n whose distance from a given half line emanating from the origin is less than or equal to r .

For every $s \in [r, R)$, let $K(s) = \{z \in S^{n-1} : sz \in \text{int}(K)\}$, where $\text{int}(K)$ denotes the interior of K , and let $C(s)$ be defined analogously. Since $rB_n \subset K$, the convexity of K implies that

$$K(r) \supset (K(s))_{\arccos(r/s)}.$$

On the other hand,

$$C(r) = (C(s))_{\arccos(r/s)}$$

and so we can write

$$\mathcal{H}_{n-1}((K(s))_{\arccos(r/s)}) \leq \mathcal{H}_{n-1}(K(r)) \leq \mathcal{H}_{n-1}(C(r)) = \mathcal{H}_{n-1}((C(s))_{\arccos(r/s)}).$$

The Brunn–Minkowski inequality on the sphere (see, for instance, [BZ], Theorem 9.1.1) states that among all sets of given volume on the sphere, the cap has the least ε -neighborhood for any ε . Therefore $\mathcal{H}_{n-1}(K(s)) \leq \mathcal{H}_{n-1}(C(s))$, for every $s \in [r, R)$, and then $\text{vol}(K) \leq \text{vol}(C)$. One can easily evaluate $\text{vol}(C)$ and obtain that $\text{vol}(C) \leq \text{vol}(\frac{R+r}{2}B_n)$. ■

Let T be a δ -net and let $t = |T|$. Let $\alpha \in (0, 1)$, $\delta > 0$, $t, f \in (0, t] \cap \mathbb{N}$ and $a > 0$ be so that

$$(5.1) \quad 1 - t^m(1 - \alpha^{2^m})^{[t/3^{m(n+1)}]} - \sum_{i=f}^t \binom{t}{i} \alpha^i (1 - \alpha)^{t-i} > 0$$

and

$$(5.2) \quad f\sigma_{n-1}(6\delta m + a) \leq n\kappa_n/2,$$

where $\sigma_{n-1}(\rho)$ denotes the $(n-1)$ -dimensional Hausdorff measure of the spherical cap in S^{n-1} with radius ρ .

As will be clear in the proof of Theorem 3, we are interested in choosing the parameters which (asymptotically in m) maximize a under the constraints (5.1) and (5.2).

LEMMA 10: *Let n be fixed. For any $\beta > 0$ it is possible to choose parameters α, δ, f and a that satisfy (5.1) and (5.2) and with*

$$a \geq m^{-2^m(1+2\beta)}, \quad \text{as } m \rightarrow \infty.$$

Proof: Choosing

$$f = \alpha t, \quad \alpha = \frac{1}{2} \left(\frac{2^m(1+\beta) - 1}{12m2^m(1+\beta)} \right)^{n-1} \quad \text{and} \quad t = \left(\frac{2m3^{m(n+1)}}{\alpha^{2^m}} \right)^{1+\beta},$$

(5.1) is satisfied, for m large enough.

Using the inequalities $\sigma_{n-1}(\rho) \leq \kappa_{n-1}\rho^{n-1}$ and (3.2), one can check that, choosing $a = m^{-2^m(1+2\beta)}$, (5.2) is satisfied too. ■

Proof of Theorem 3: Let T be a δ -net and $t = |T|$. Let $\alpha \in (0, 1)$, $\delta > 0$, $t, f \in (0, t] \cap \mathbb{N}$ and $a > 0$ be parameters that satisfy (5.1) and (5.2). By Lemma 6 and Lemma 7, there exists a subset A' of T such that $(A')_{6m\delta}$ satisfies the m -orbit property, with $|A'| \leq f$. Let H_m be the convex hull of rS^{n-1} and of $R(A')_{6m\delta}$, with $r > 0$ and $R > 0$ chosen so that $r/R = \cos a$ and moreover $\text{vol}(H_m) = \kappa_n$. By construction the outer radius of H_m is not decreased by any m symmetrizations.

We claim that $\mathcal{H}_{n-1}(\partial H_m \cap rS^{n-1}) \geq n\kappa_n r^{n-1}/2$. By the relation $r/R = \cos a$ it follows easily that every point of $rS^{n-1} \setminus r(A')_{6m\delta+a}$ is contained in the boundary of H_m . Hence by (5.2) we deduce

$$\begin{aligned} \frac{\mathcal{H}_{n-1}(\partial H_m \cap rS^{n-1})}{r^{n-1}} &\geq n\kappa_n - \mathcal{H}_{n-1}((A')_{6m\delta+a}) \\ &\geq n\kappa_n - f\sigma_{n-1}(6m\delta + a) \geq n\kappa_n/2. \end{aligned}$$

We can now apply Lemma 9 to conclude that the Hausdorff distance from H_m to B_n is attained by the outer sphere. Using Lemma 10 this distance can be estimated, as $m \rightarrow \infty$, by

$$d_H(H_m, B_n) \geq \frac{R-r}{2} = \frac{R}{2}(1 - \cos a) \geq \frac{a^2}{6} \geq \frac{1}{6m^{2(1+2\beta)2^m}}.$$

Let us denote $d_H(H_m, B_n)$ by $\varepsilon(n, m)$. By definition of $N(n, \varepsilon)$, it is certainly true that $N(n, \varepsilon(n, m)) \geq m$. Expressing m in terms of $\varepsilon(n, m)$ one gets that, as $m \rightarrow \infty$,

$$N(n, \varepsilon(n, m)) \geq \frac{\log(\log(\frac{1}{\varepsilon(n, m)}))}{\log(2)}(1 + o(1)).$$

Since N is monotone, this concludes the proof. ■

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